Lecture 1.

prime divisor, Weil divisor, effective Weil divisor, divisor of f,

principal Weil divisor, linear equivalence, class group

Cartier divisor, local equations, principal Cartier, effective Cartier

$\mathrm{Div} \hookrightarrow \mathrm{WDiv}$ for normal vties

Lecture 2.

$\mathcal{O}_X(D)$

Cartier divisors $\xleftrightarrow{\sim}$ invertible subsheaves of $\mathcal{K}_X$

$\mathrm{Div}/\sim \hookrightarrow \mathrm{Pic}$, iso on integral / projective schemes

$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$

$(m_1, \dots, m_r) \mapsto \chi(X, m_1 D_1 + \dots + m_r D_r)$ polynomial

intersection number

Intersection numbers: multilinear symmetric, integral, $D_1 \cdots D_n = (D_2 \cdots D_n)_{Y_1}$

$D_1 \cdots D_n = \#(D_1 \cap \dots \cap D_n)$ when X smooth, D_i meet transversally

Projective formula: $\pi^*(D_1) \cdots \pi^*(D_n) = \deg(\pi)\, D_1 \cdots D_n$ when $\pi : Y \twoheadrightarrow X$,

Y, X proper, $n \geq \dim Y$.

globally generated sheaf

$f : X \to \mathbb{P}^N \Rightarrow f^* \mathcal{O}_{\mathbb{P}^N}(1)$ glob gen inv'ble sheaf, gen: $f^* x_0, \dots, f^* x_N$

If $\mathcal{L}$ inv'ble, $s_1, \dots, s_N \in \Gamma(X, \mathcal{L})$ are generators $\Rightarrow \exists!\, f : X \to \mathbb{P}^N$ s.t.

$\mathcal{L} = f^* \mathcal{O}_{\mathbb{P}^N}(1)$, $s_i = f^* x_i$

divisor of zeroes

$\forall s \in H^0(X, \mathcal{O}_X(D_0))$: $(s)_0 \sim D_0$, $(s)_0$ eff div; every $D \sim D_0$ effective is

of this form; $(s)_0 = (s')_0 \Leftrightarrow s' = \lambda s$ for some $\lambda \in k$

(complete) linear system, base ideal, base locus, $\varphi_{|V|}$, bpf

bpf $\Leftrightarrow$ globally generated for global sections

$\{\text{nondeg rat maps } X \dashrightarrow \mathbb{P}^n\} / \text{proj equivalence} \longleftrightarrow \{\text{lin systems in dim } n\} / \text{iso}$

(very) ample

CSG: $\mathcal{L}$ ample $\iff$ $\forall \mathcal{F}$ coh: $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ $\forall i>0$ $\forall m \gg 0$

$\iff$ $\forall \mathcal{F}$ coh: $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ glob gen $\forall m \gg 0$

$\iff$ $\mathcal{L}^{\otimes m}$ very ample $\forall m \gg 0$

Lecture 3.

On curves: ample $\iff$ $\deg > 0$.

Nakai–Moishezon: X proper k-scheme, $D \in \mathrm{Div}(X)$. Then

D ample $\iff$ $\forall V \subseteq X$ integral: $D^{\dim V} . V > 0$

$f: X \to Y$ fin, X, Y proper, $\mathcal{L}$ ample $\Rightarrow$ $f_* \mathcal{L}$ ample

X proper k-sch: $\mathcal{L}$ ample $\iff$ $\mathcal{L}|_{X_{red}}$ ample $\iff$ $\mathcal{L}|_Z$ ample $\forall Z \subseteq X_{red}$ irred

$f: X \twoheadrightarrow Y$ fin, X, Y proper k-schemes. $\mathcal{L}$ lb on X, $f^* \mathcal{L}$ ample $\Rightarrow$ $\mathcal{L}$ ample

$\mathrm{Div}(X)_{\mathbb{Q}}$, $\mathrm{Div}(X)_{\mathbb{R}}$, nef

D nef $\Rightarrow$ $f^* D$ nef; $f^* D$ nef & f surj $\Rightarrow$ D nef

X proper $/k$, D nef $\Rightarrow$ $\forall V \subseteq X$ integral: $D^{\dim V} . V \geq 0$.

X proper $/k$, D nef, H ample $\Rightarrow$ $\forall \underset{\in \mathbb{Q}}{\varepsilon > 0}$: $D + \varepsilon H$ ample

Lecture 4.

$Z_1(X)$, numerical equivalence, N^1, N_1, ρ

Severi: $N^1(X)$ fin gen

$N^1(X) \times N_1(X) \longrightarrow \mathbb{Z}$ perfect pairing

$NE(X)$, $\overline{NE}(X)$

Kleiman: X proj $/k$, $D \in \mathrm{Div}(X)$ ample $\iff$ $\forall z \in \overline{NE}(X) \setminus \{0\}$: $D.z > 0$

$\overline{NE}(X)$ contains no line

$\mathrm{Amp}(X)$, $\mathrm{Nef}(X)$, extremal subcone / face / ray

$\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$, $\mathrm{int}\, \mathrm{Nef}(X) = \mathrm{Amp}(X)$, $\overline{NE}(X) = \left\{ z \in N_1(X) \;\middle|\; \begin{array}{l} z.\delta \geq 0 \\ \forall \delta \in \mathrm{Nef}(X) \end{array} \right\}$

Proj morph f of noetherian schemes, $f_* \mathcal{O}_X = \mathcal{O}_Y$ $\Rightarrow$ $\forall f^{-1}(y)$ conn.

ZMT: $f: X \to Y$ proper birat, X, Y integral, Y normal $\Rightarrow$ $f_* \mathcal{O}_X = \mathcal{O}_Y$

Stein: $f: X \to Y$ proj, X, Y noeth, $\Rightarrow$ $f = X \xrightarrow{f'} Y' \xrightarrow{\pi} Y$, π finite,

$Y' = \underline{\mathrm{Spec}}\, f_* \mathcal{O}_X$, $f'_* \mathcal{O}_X = \mathcal{O}_{Y'}$

$NE(\pi)$

$NE(\pi) \subseteq NE(X)$ is extremal

Rigidity Lemma: $\pi_* \mathcal{O}_X = \mathcal{O}_Y$, $\pi' : X \to Y'$, $NE(\pi) \subseteq NE(\pi')$

$$\Rightarrow \quad \begin{tikzcd} X \arrow[r, "\pi'"] \arrow[dr, "\pi"'] & Y' \\ & Y \arrow[u, dotted, "\exists f'"'] \end{tikzcd}$$

When $\pi_* \mathcal{O}_X = \mathcal{O}_Y$, $NE(\pi)$ determines π.

Examples. Mumford's example.

Lecture 5.

$\mathrm{Exc}(f)$

Negativity lemma: $f : Y \to X$ proper birat, Y, X normal, $-D$ f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Then D eff $\Leftrightarrow$ $f_* D$ eff

$\mathbb{Q}$-factorial

van der Waerden purity: $f : Y \to X$ proper birat, Y, X normal, X $\mathbb{Q}$-factorial $\longrightarrow$ $\forall Z \subseteq \mathrm{Exc}(f)$ irred: $\operatorname{codim} Z = 1$

Ramification formula: $f : Y \to X$ proper birat, Y, X smooth $\Rightarrow$ $K_Y \sim f^* K_X + R$, R effective, $\operatorname{Supp} R = \mathrm{Exc}(f)$

Lecture 6.

$\pi : Y \to X$ proper birat, X, Y normal

- $D \in \operatorname{Div} X$, $F \in \operatorname{Div} Y$ eff, $F \subseteq \mathrm{Exc}(\pi)$ $\Rightarrow$ $H^0(X, D) \cong H^0(Y, \pi^* D + F)$, i.e. $\pi_* \mathcal{O}_Y(F) = \mathcal{O}_X$
- Y smooth, $D, D' \in \operatorname{Div} X$, $F', F \in \operatorname{Div} Y$, $F, F' \subseteq \mathrm{Exc}(\pi)$,

  $\pi^* D + F \sim \pi^* D' + F'$ $\Rightarrow$ $D \sim D'$ & $F = F'$

terminal / canonical singularities

smooth $\Rightarrow$ terminal $\Rightarrow$ canonical

suffices to check only one resolution

Terminal $\Leftrightarrow$ $f : Y \to X$ resol, $mK_X \in \operatorname{Div} X$ $\Rightarrow$ $f_* \mathcal{O}_Y(mK_Y - E) \cong \mathcal{O}_X(mK$

$\forall E$ exceptional reduced divisor

In dimension 2, terminal singularities are smooth.

Lecture 7.

$k(D)$ Iitaka dimension, $S(D)$ semigroup, $e(D)$ exponent, big divisor

X normal complete vty, $D \in \operatorname{Div} X \Rightarrow \exists a, b > 0$:

$$a m^{k(D)} \leq h^0(X, mD) \leq b\, m^{k(D)} \quad \forall m \in S(D),\ m \gg 0$$

Kodaira's Lemma: $D \in \operatorname{Div} X$ big, $F \in \operatorname{Div} X$ effective $\Rightarrow h^0(X, mD - F) \neq 0$ $\forall m \in S(D), m \gg 0$

X proper normal vty, $D \in \operatorname{Div} X$. Then D big $\Leftrightarrow$

$\Leftrightarrow \forall A \in \operatorname{Div} X$ ample: $mD \sim A + E$ for some $m > 0$, $E \geq 0$

$\Leftrightarrow$ $\varphi_{|mD|}$ is birat for $m \in S(D)$, $m \gg 0$

$k(X, K_X)$ Kodaira dimension,

of general type

canonical ring $R(X, K_X)$

X_{can} canonical model

BCHM (2010): X sm proj vty $\Rightarrow R(X, K_X)$ fin gen

X proper normal vty, $D \in \operatorname{Div} X$, $H^0(X, D)$ generates $\bigoplus_{m \geq 0} H^0(X, mD)$,

$Z = \operatorname{Im} \varphi_{|D|}(X) \subseteq \mathbb{P}^N$, H a hyperplane section, D big. Then:

φ birat, Z normal, $\forall D' \in \operatorname{Bs}|D|$ is contracted by φ,

$\varphi_* D \sim H$, if D is nef then $|D|$ is bpf

D big + nef, $R(X, D)$ fin gen $\Rightarrow$ $|D|$ bpf

Reid (1980): X smooth proper vty of gen type, $R(X, K_X)$ fin gen. Then:

X_{can} is a normal proj vty, X_{can} birat to X

$K_{X_{can}}$ is $\mathbb{Q}$-Cartier ample, X_{can} has canon sings,

if K_X is nef then $X \dashrightarrow X_{can}$ is a morphism

SNC

log resolution

log pair

boundary

Lecture 8.

f_*^{-1} strict transform

$a(E, X, \Delta)$ discrepancy wrt. (X, Δ)

klt, lc, discrepancy, total discrepancy

$a(E, X, \Delta) = k - 1 - \sum a_i \, \mathrm{mult}_{Z_i}(D_i)$ for $\pi : Bl_Z X \to X$, $E \subset \mathrm{Exc}\,\pi$, $\Delta = \sum a_i D_i$

X smooth, $Z \subseteq X$ closed smooth, $\mathrm{codim}(Z, X) = k$

$a(E, X, \Delta_X) = a(E, Y, \Delta_Y)$ for $f : Y \to X$ bir proper, Δ_Y, Δ_X $\mathbb{Q}$-div,

$(X, \Delta_X), (Y, \Delta_Y)$ log pairs, $K_Y + \Delta_Y = f^*(K_X + \Delta_X)$ ~~[illegible]~~

X smooth, $\sum D_i$ SNC, $\Delta = \sum a_i D_i$

$\Rightarrow \mathrm{discr}(X, \Delta) = \min\left(\min\left\{ 1 - a_i - a_j \mid i \neq j,\ D_i \cap D_j \neq \emptyset \right\}, \min\{1 - a_i \mid i\}, 1\right)$

(X, Δ) lp, $\Delta \geq 0$, $f : Y \to X$ log resol, $K_Y \sim f^*(K_X + \Delta) + \sum a_i(E) \cdot E$

Then (X, Δ) klt $\Leftrightarrow a_i(E) > -1$, lc $\Leftrightarrow a_i(E) \geq -1$

$\mathrm{discr} = -\infty$ OR $-1 \leq \text{tot discr} \leq \mathrm{discr} \leq +1$

lc $\Rightarrow 0 \leq a_i \leq 1$, lc $\Leftrightarrow \mathrm{discr} \geq -1$ for Δ eff, klt $\Leftrightarrow \mathrm{discr} > -1$ & $\left\{ \lfloor \Delta \rfloor \leq 0 \right.$

X sm, $\sum D_i$ SNC, $a_i \geq 0$. Then $(X, \sum a_i D_i)$ lc $\Leftrightarrow a_i \leq 1$,
klt $\Leftrightarrow a_i < 1$

Cone over ell curve: lc but not klt.

multiplier ideal sheaf $\mathcal{J}((X, \Delta), D) = \mathcal{J}(X, \Delta + D)$

$\Delta, D \geq 0 \Rightarrow \mathcal{J}(X, \Delta + D) \subseteq \mathcal{O}_X$. (X, Δ) klt $\Leftrightarrow \mathcal{J} = \mathcal{O}_X$.
lc $\Leftrightarrow \mathcal{J}(X, (1-\varepsilon)\Delta) = \mathcal{O}_X \quad \forall \varepsilon \in (0, 1]$

non-klt locus

X sm, D $\mathbb{Q}$-div with SNC support, $f : Y \to X$ log res of (X, D)

$\Rightarrow f_* \mathcal{O}_Y\left(K_Y - \lfloor f^*(K_X + D) \rfloor\right) = \mathcal{O}_X\left(-\lfloor D \rfloor\right)$

X sm, $D \geq 0$ $\mathbb{Q}$-div, $\mathrm{mult}_x D \geq \dim X$ for some $x \in X$

$\Rightarrow \mathcal{J}(X, D)_x \subseteq \mathfrak{m}_x \subsetneq \mathcal{O}_X$

Lecture 9.

KVT: X smooth proj $/\mathbb{C}$, A ample lb $\Rightarrow H^i(X, K_X+A)=0 \ \forall i>0$

KVVT: X smooth proj, L nef big $\mathbb{Q}$-div, $\lceil L\rceil - L$ with SNC supp

$\Rightarrow H^i(X, K_X+\lceil L\rceil)=0 \ \forall i>0$

Spec. case: L Cartier (integral) $\Rightarrow H^i(X, K_X+L)=0 \ \forall i>0$

Local Vanishing Lemma: $f: Y\to X$ log res of (X,Δ)

$\Rightarrow R^i f_* \mathcal{O}_Y\left(K_Y - \lfloor f^*(K_X+\Delta)\rfloor\right)=0 \quad \forall i>0$

Nadel Vanishing: (X,Δ) log pair, $L\in \operatorname{Div} X$, $L-(K_X+\Delta)$ nef & big

$\Rightarrow H^i\left(X, \mathcal{J}(X,\Delta)\otimes \mathcal{O}_X(L)\right)=0 \quad \forall i>0$

(X,Δ) klt $\Rightarrow \mathcal{J}(X,\Delta)=\mathcal{O}_X \Rightarrow H^i(X,L)=0 \quad \forall L = K_X+\Delta+(\text{nef\&big})$

centre $c_X(E)$

log canonical centre $\Rightarrow \mathcal{J}(X,\Delta)\subseteq I_W$

isolated lc centre

(X,Δ) lc, W isol lc centre $\Rightarrow I_W = \mathcal{J}(X,\Delta)$

Shokurov NVT: (X,Δ) klt, $L\in\operatorname{Div} X$ nef, $pL-(K_X+\Delta)$ nef

and big for some $p>0$. $\Rightarrow |dL|\neq \emptyset \quad \forall d\gg 0$.

Kawamata subadjunction: (X,Δ) lc, W isol lc centre,

A ample divisor $\Rightarrow (K_X+\Delta+\varepsilon A)|_W \sim_{\mathbb{Q}} K_W+\Delta_W \quad \forall \varepsilon>0$ for (W,Δ_W) klt

Tie breaking: (X,Δ) klt, D $\mathbb{Q}$-div, $(X,\Delta+D)$ log canonical,

W minimal lc centre for $(X,\Delta+D)$, A ample

$\Rightarrow \exists\, 0<\varepsilon,\eta\ll 1$, $\varepsilon,\eta\in\mathbb{Q}$, $\exists D'\sim_{\mathbb{Q}} A$: W isol lc centre for $\left(X, \Delta+(1-\varepsilon)D+\eta D'\right)$

Lecture 10.

Bpf Thm: (X,Δ) klt, L nef Cartier, $pL-(K_X+\Delta)$ nef big, $p\in\mathbb{Q}_{>0}$

$\Rightarrow |mL|$ bpf $\forall m\gg 0$.

(X,Δ) klt, $|D|$ lin sys, $\dim|D|>0$, $S\in|D|$, $c:=\inf\{t\in\mathbb{Q} \mid K_X+\Delta+tS \text{ not klt}\}$

- $c<1 \Rightarrow \forall W$ lc centre of $(X,\Delta+cS)$: $W\subseteq \operatorname{Bs}|D|$
- $c=1 \Rightarrow$ either $\exists W$ lc centre s.t. $W\subseteq \operatorname{Bs}|D|$

or S reduced with irreducible conn components,

and these components are the lc centres of $(X,\Delta+cS)$

(X,Δ) klt, $K_X+\Delta$ nef big $\Rightarrow$ $K_X+\Delta$ is semiample

X terminal, K_X nef big $\rightarrow$ K_X semiample, $\varphi_{|I m K_X|}: X \to X^{can}$,

X^{can} has can singularities and $K_{X^{can}}$ is ample

Abundance Conj: $\cdot$ (X,Δ) lc, $K_X+\Delta$ nef $\Rightarrow$ $K_X+\Delta$ semiample;

$\cdot$ X terminal, K_X nef $\Rightarrow$ K_X semiample

Zariski's example

Rationality Thm: (X,Δ) klt, $K_X+\Delta$ not nef, $a\in\mathbb{Z}_{>0}$, $a\cdot(K_X+\Delta)\in \operatorname{Div} X$,

$H\in \operatorname{Div} X$ nef big, $r:=\sup\{t\in\mathbb{Q} \mid H+t(K_X+\Delta) \text{ nef}\}$ nef threshold

$\Rightarrow$ $r\in\mathbb{Q}$, $r=\frac{u}{v}$, $v\le a\cdot(\dim X+1)$

$P(x,y)$ polynomial, $\deg P\le n$, $a\in\mathbb{Z}_{>0}$, $\varepsilon\in\mathbb{R}_{>0}$, $\exists r\in\mathbb{R}$:

$P(x,y)=0$ $\forall x,y\in\mathbb{N}$, $0<ay+rx<\varepsilon$.

$\Rightarrow$ $r\in\mathbb{Q}$, $r=\frac{u}{v}$, $v\le \frac{a\cdot(n+1)}{\varepsilon}$

Lecture 11.

Cone Thm.: (X,Δ) projective klt pair. Then there are countably many $(K_X+\Delta)$-negative extremal rays $R_i\subseteq\overline{NE}(X)$ s.t.

$\overline{NE}(X)=\overline{NE}_{(K_X+\Delta)\ge 0}+\sum R_i$, $R_i=\overline{NE}\cap L_i^{\perp}$,

$\forall H$ ample $\forall \varepsilon>0$ $\overline{NE}(X)=\overline{NE}_{K_X+\Delta+\varepsilon H\ge 0}+\sum_{fin} R_i$

$\forall R_i$ is generated by a rational curve

Contraction Thm: (X,Δ) projective klt pair, R $(K_X+\Delta)$-negative extremal ray $\Rightarrow$ $\exists!\ \varphi: X\to Z$, Z is projective normal vty,

$\varphi_*\mathcal{O}_X=\mathcal{O}_Z$, $\varphi(C)=pt$ for $C\subseteq X$ curve iff $[C]\in R$;

$D\in \operatorname{Div} X$, $D.R=0$ $\Rightarrow$ $D=\varphi^* D_Z$ for some $D_Z\in\operatorname{Div} X$.

(X,Δ) projective klt pair, R $(K_X+\Delta)$-negative extremal ray,

$[C]\in R$, $\varphi: X\to Z$ contraction $\Rightarrow$ $0\to \operatorname{Pic} Z \xrightarrow{\varphi^*} \operatorname{Pic} X \to \mathbb{Z}$
$L \mapsto L.C$

In partic: $\rho(Z)=\rho(X)-1$.

Lecture 12.

X normal proj vty, $\mathbb{Q}$-factorial, $f: X \to Y$ contraction of an extremal ray $R \subseteq \overline{NE}(X)$. Then one of the following holds:

- $\dim Y < \dim X$ — fibre type
- $Ex(f)$ irred divisor & f birat — divisorial
- $\operatorname{codim} Ex(f) \geq 2$ & f birat — small

X proj vty with terminal / klt singularities, $f: X \to Y$ contraction of a K_X-neg extr ray, X $\mathbb{Q}$-factorial, f divisorial / of fibre type.
$\Rightarrow$ Y $\mathbb{Q}$-factorial.

f small contraction, X klt $\Rightarrow$ K_Y not Cartier

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

X, X', Y proper normal varieties, f, f' proper normal,
$-K_X$ $\mathbb{Q}$-Cartier, f nef, $K_{X'}$ $\mathbb{Q}$-Cartier, f' nef
$\Rightarrow$ $\forall E$ exc div $/Y$: $a(E,Y) \leq a(E,Y')$

X normal proj $\mathbb{Q}$-factorial, $f: X \to Y$ divisorial contraction of a K_X-neg extr ray. X term/can/klt $\Rightarrow$ Y term/can/klt.

Flip

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

flip.

- X term $\Rightarrow$ X^+ term
- X $\mathbb{Q}$-fac $\Rightarrow$ X^+ $\mathbb{Q}$-fac
- If the flip of f exists, it is unique, $X^+ = \operatorname{Proj} \bigoplus f_* \mathcal{O}_X(mK_X)$

Mori: existence of flips in dim 3, the process terminates

MMP works in dim 4 (uses difficulty)

Flips exist for klt.

X smooth $\Rightarrow$ $\bigoplus H^0(X, mK_X)$ fin gen

X of gen type $\Rightarrow$ MMP.

difficulty